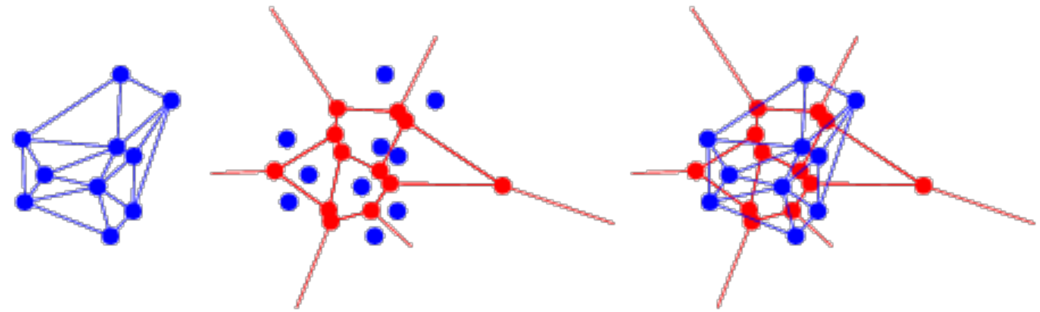
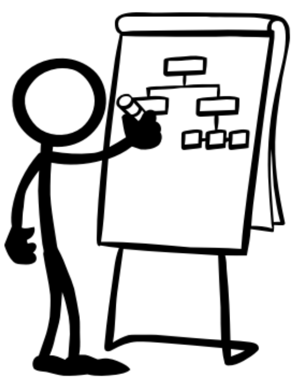
Voronoi Diagram



Delaunay triangulation *Voronoi diagram* *Delaunay and Voronoi*



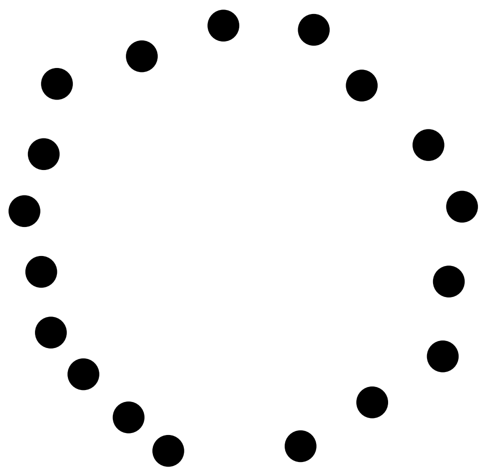
Amani Shhadi

Based on slides by
Yufei Zheng - 郑羽霈

How round is an object?

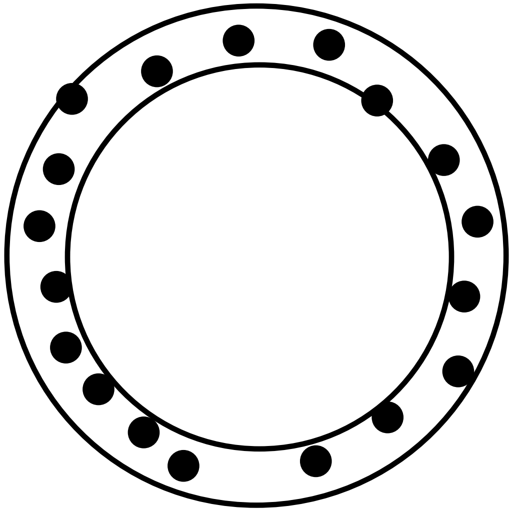
How round is an object?

- Formal problem:
- Given samples from the surface of a quasi-circular object, we would like to quantify how round it is.



Smallest width ring

- We can come up with many measures
- We will consider the following measure: What is the width of the minimal ring that contain all the samples?

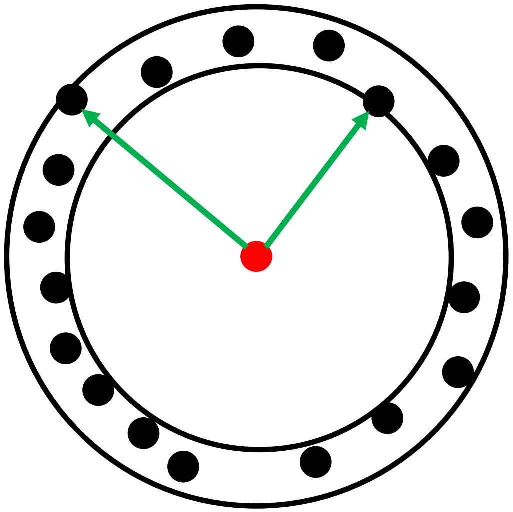


Smallest width ring

We assume that the two circles has the same center

Observations:

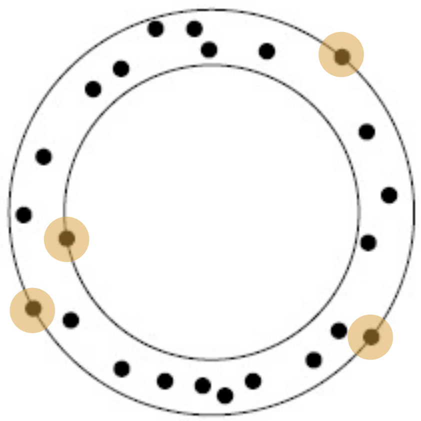
- It suffice to find the center of the ring



Smallest width ring

Observations:

- The rings are determined by 4 points

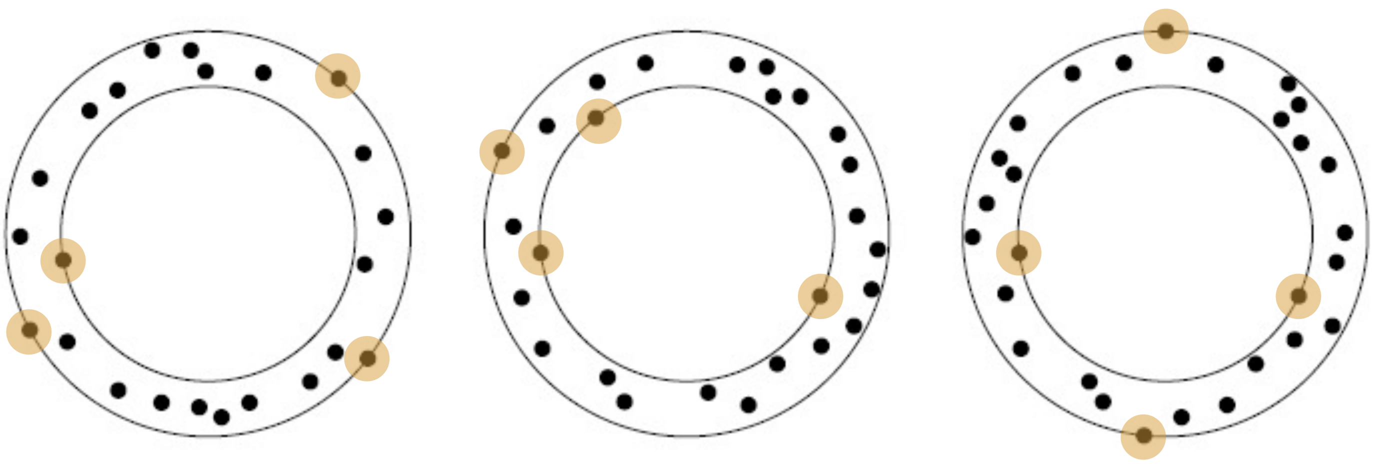


Case 1:
3 outer 1 inner

Smallest width ring

Observations:

- The rings are determined by 4 points

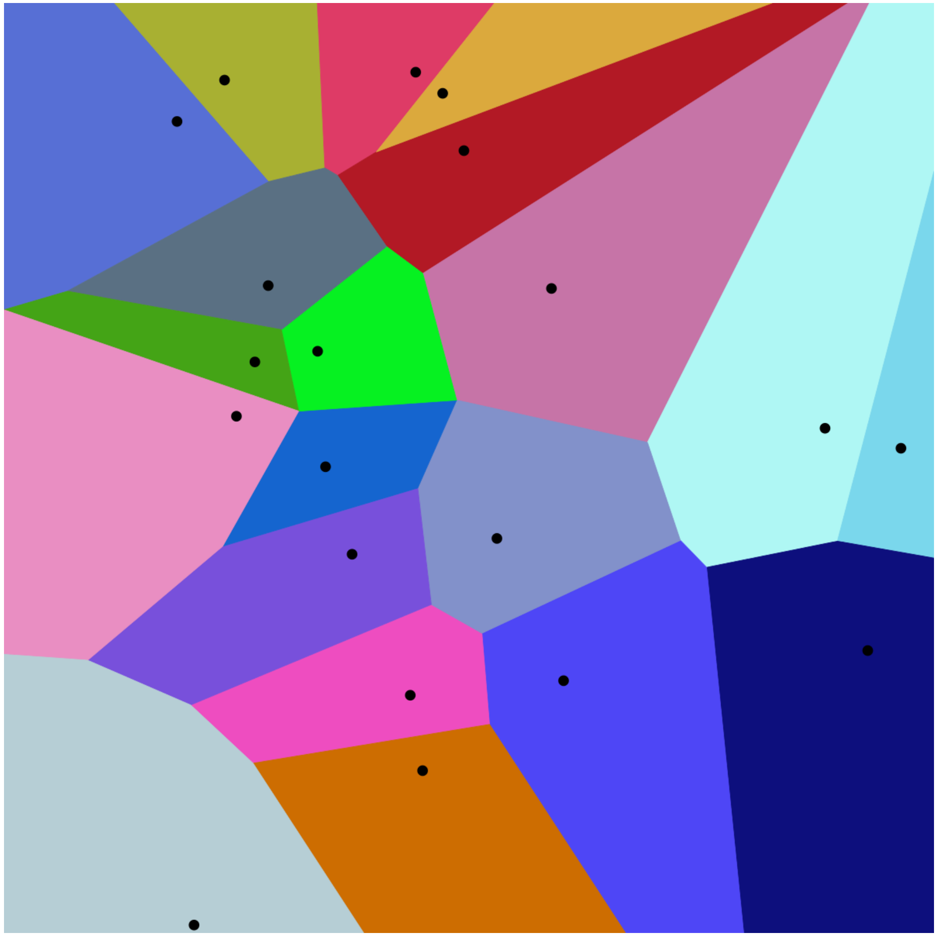


Case 1:
3 outer 1 inner

Case 2:
1 outer 3 inner

Case 3:
2 outer 2 inner

Ordinary Voronoi Diagram



Ordinary Voronoi Diagram - Recall

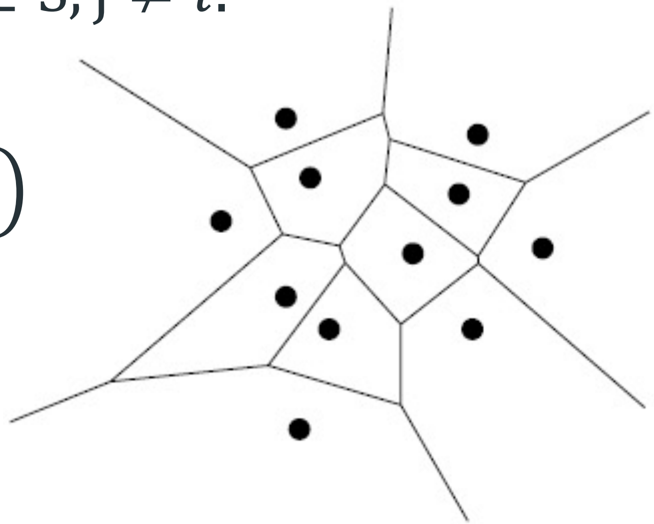
- **Definition** – a subdivision of plane into cells
  - Sites: $S = \{s_1, s_2, \cdots, s_n\}$
  - Euclidean distance in the plane

$$\text{dist}(p,q) = \sqrt{(p_x - q_x)^2 + (p_y - q_y)^2}.$$

  - p lies in the cell of site s_i iff

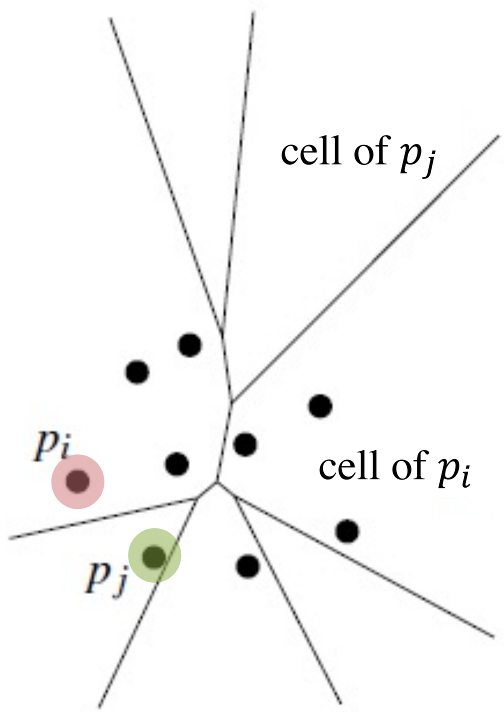
$$\text{dist}(p, s_i) < \text{dist}(p, s_j), \forall s_j \in S, j \neq i.$$

- Cells - $V(s_i) = \cap_{1 \le j \le n, j \ne i} h(s_i, s_j)$
- Edges - straight line segments



Farthest point Voronoi diagram

- Each cell is associated with the **farthest** point from the cell



Farthest point Voronoi diagram

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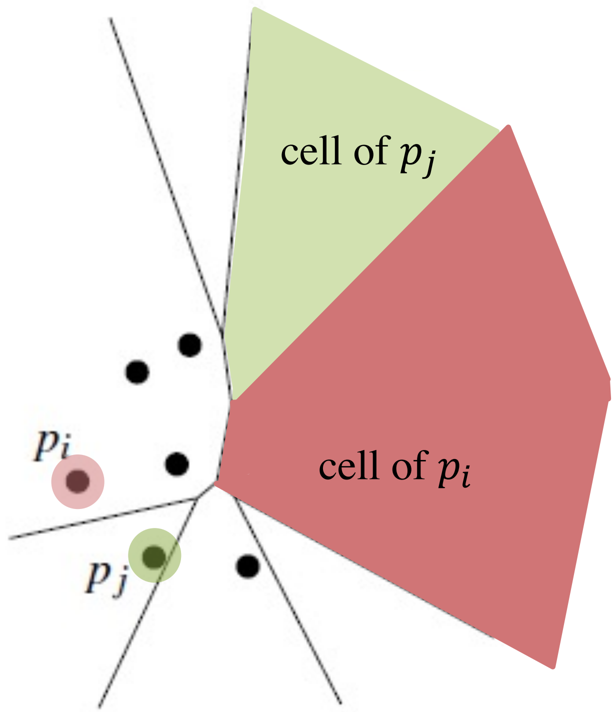


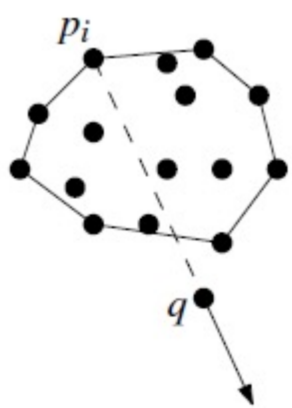
Farthest point Voronoi diagram

Observations:

- The diagram is the intersection of the "Other side" of the bisector half-planes.

- A point p has a cell iff p is a vertex of the convex hull of the point.

- If the farthest point from q is p_i, then, the ray from q in the opposite direction to p_i is also in the cell
  ⇒ The cells are unbounded.

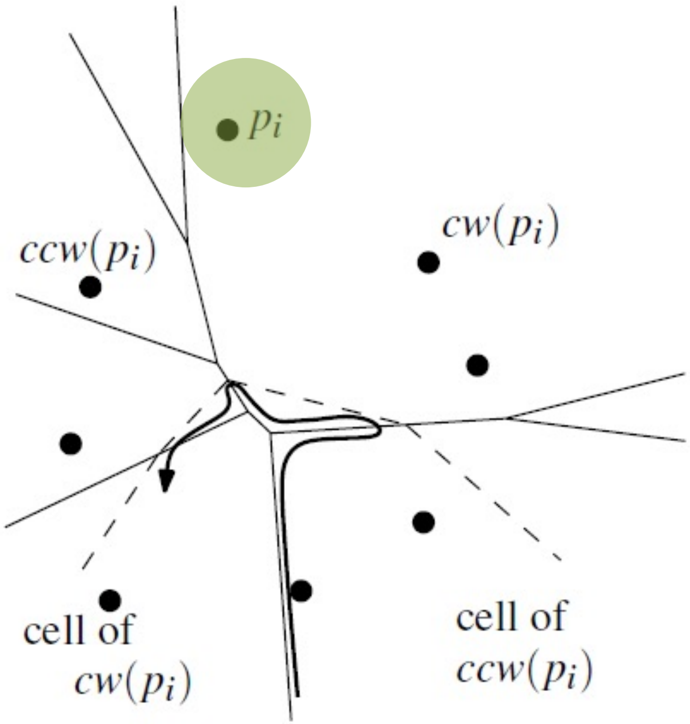


- The separator between the cells of p_i and p_j is the bisector of p_i and p_j

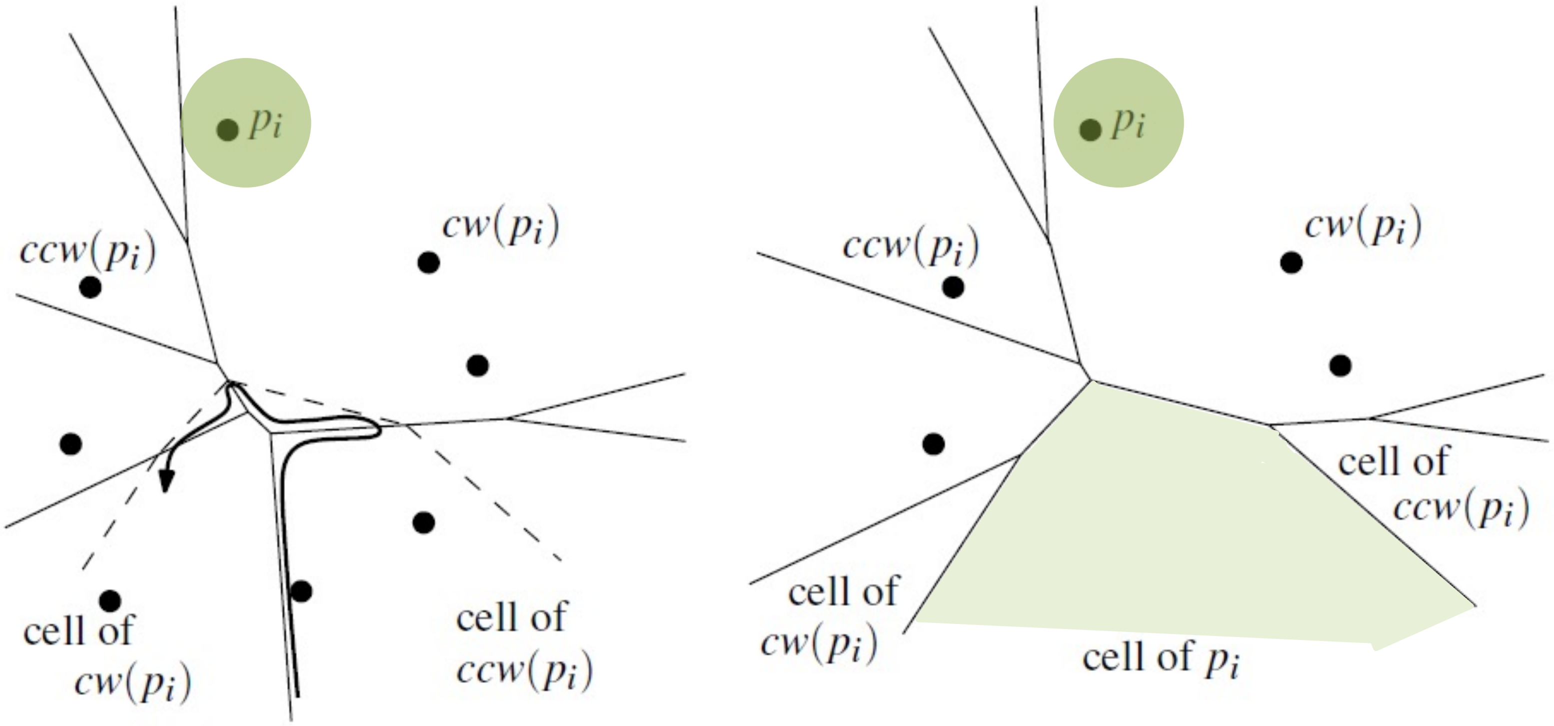
Farthest point Voronoi diagram

- Consider a random order of the CH vertices, $p_1, \dots, p_h$
- Given a diagram for $p_1, \dots, p_{i-1}$ we would like to add p_i
- We will denote the neighbors of p_i (when p_i is added) as $cw(p_i)$ and $ccw(p_i)$
- How do we find $cw(p_i)$ and $ccw(p_i)$?
  - Remove the points in the opposite order, the neighbors when p_i is removed are $cw(p_i)$ and $ccw(p_i)$

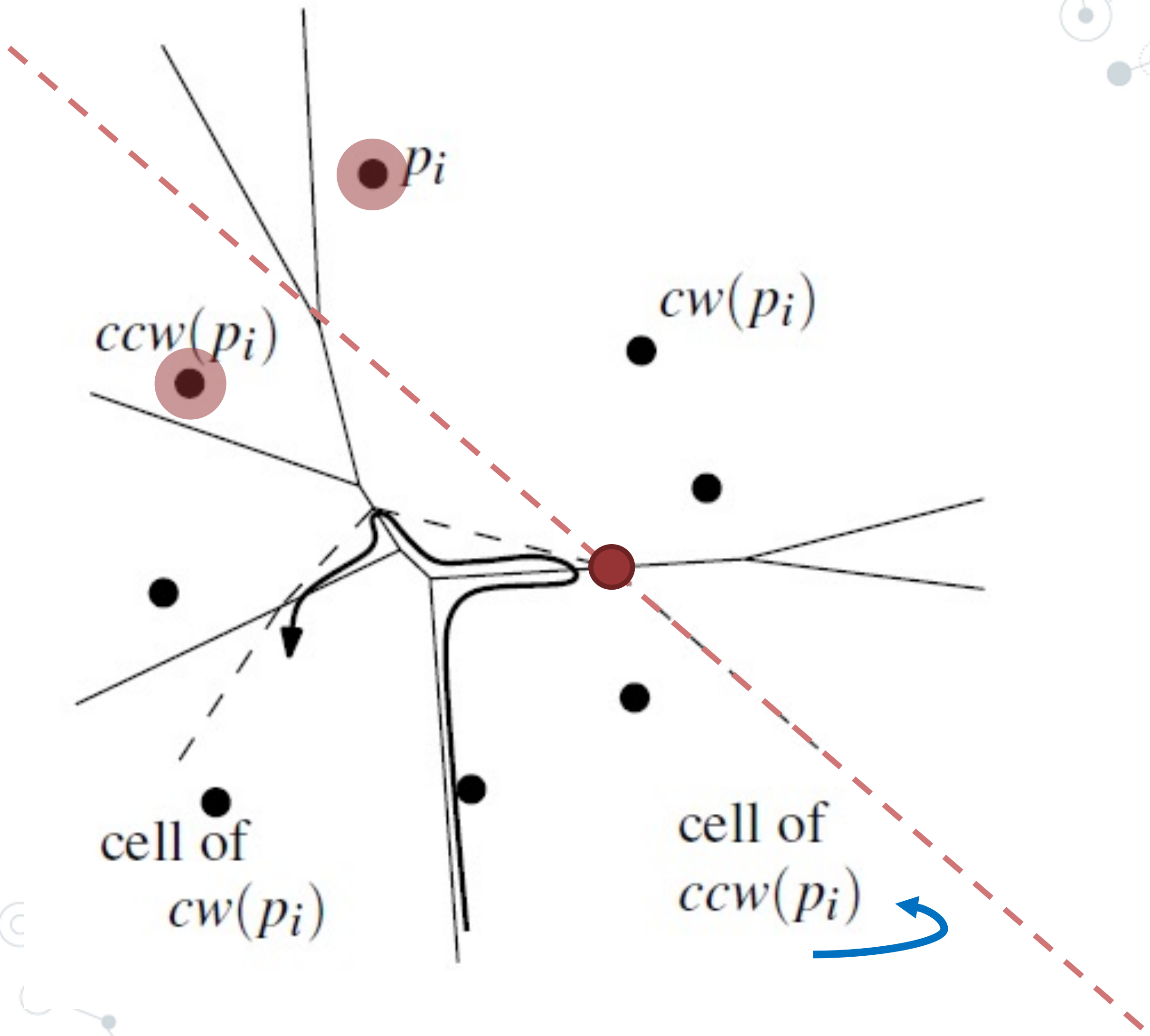
Farthest point Voronoi diagram



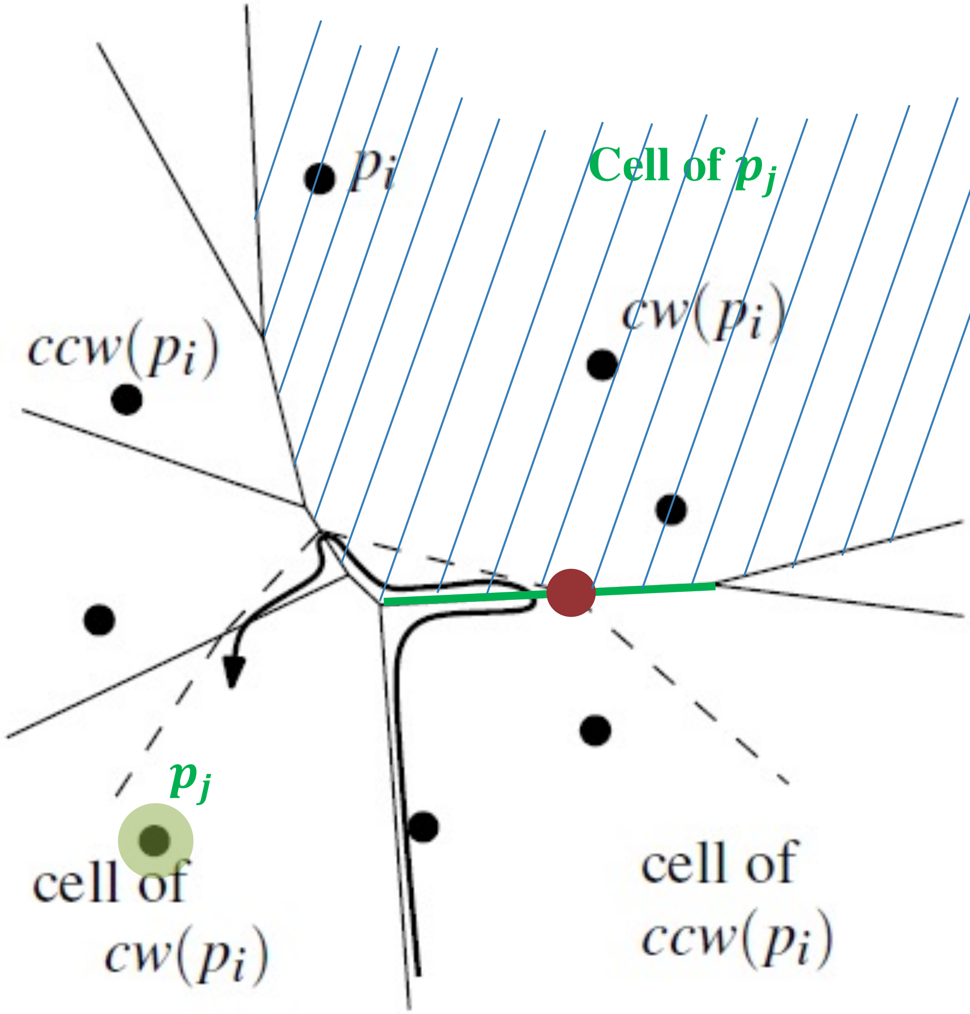
Farthest point Voronoi diagram



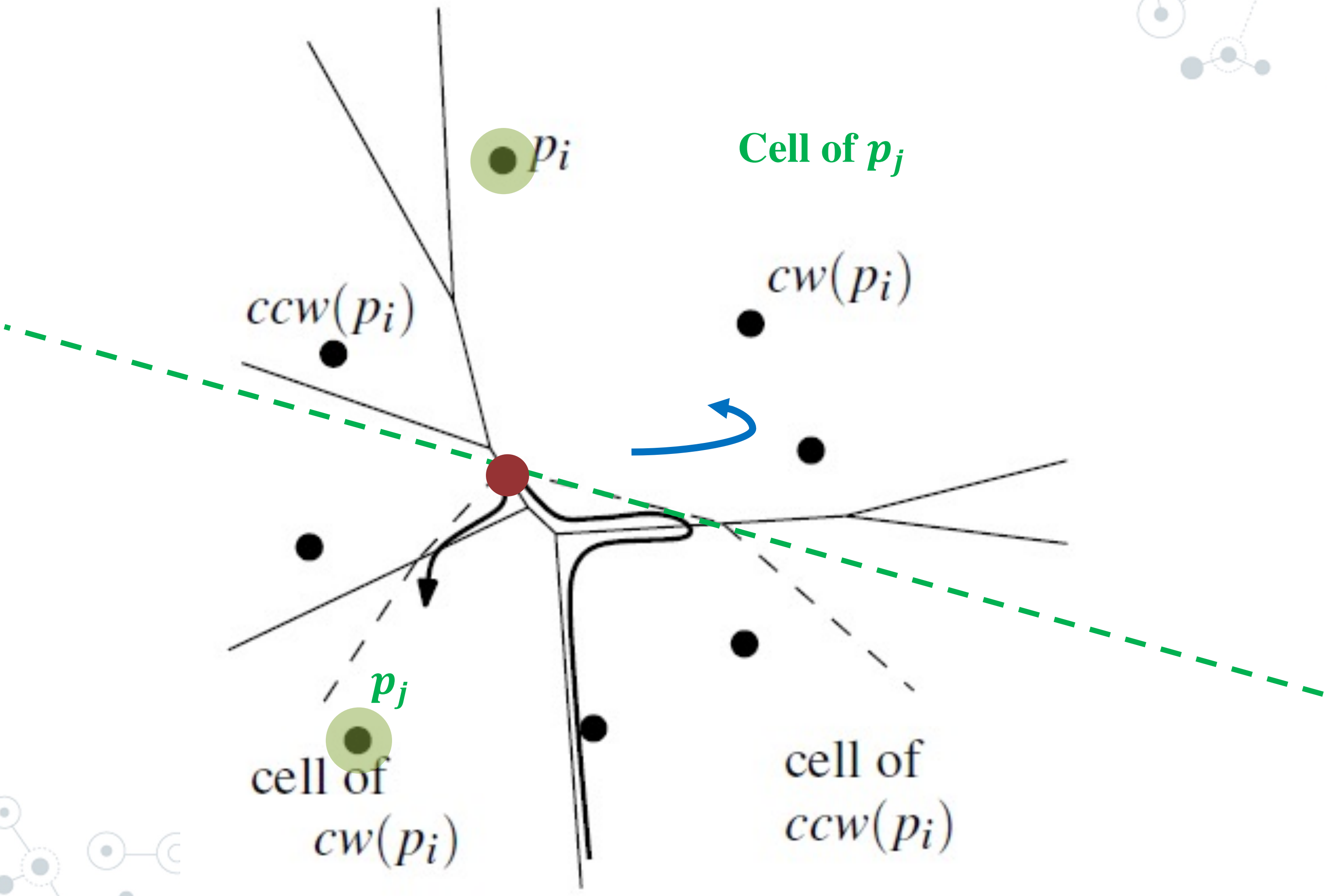
Farthest point Voronoi diagram



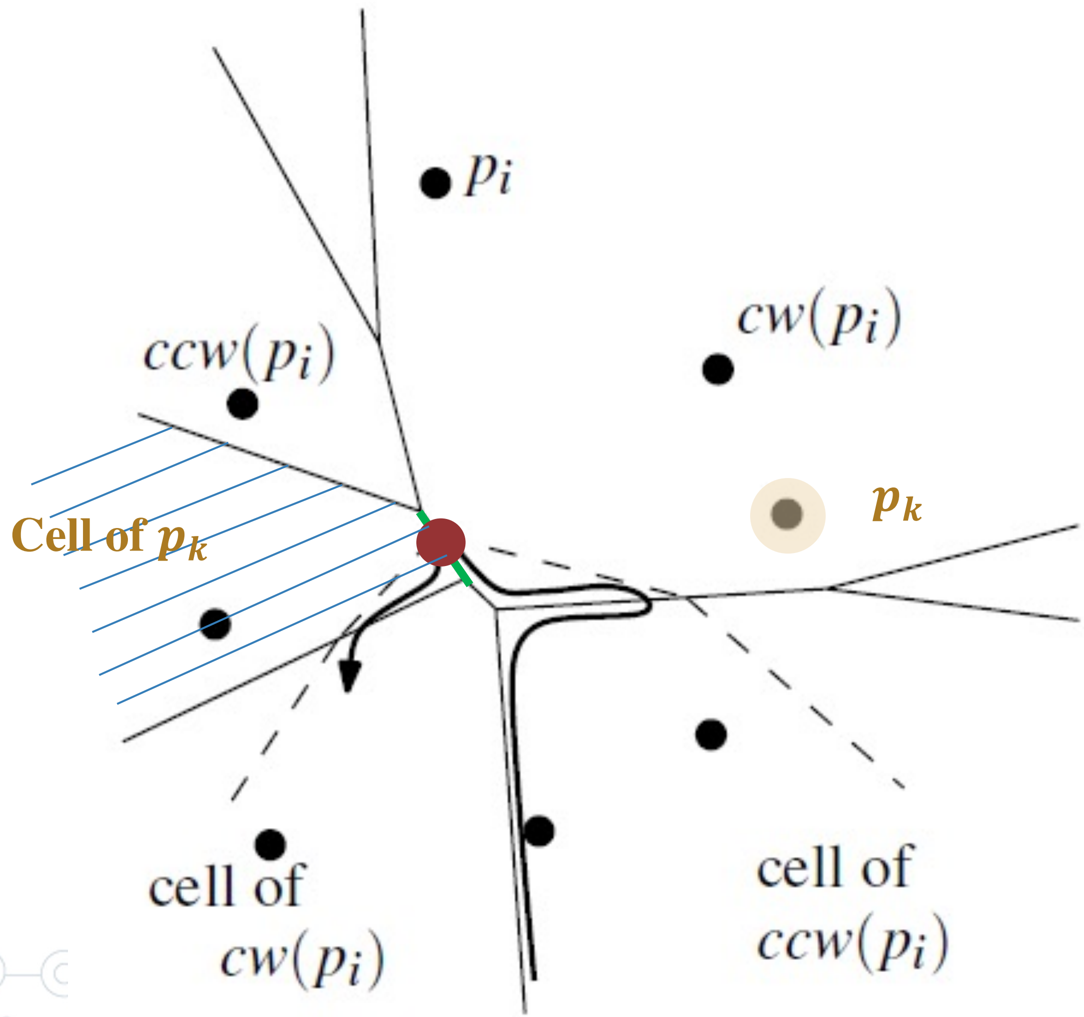
Farthest point Voronoi diagram



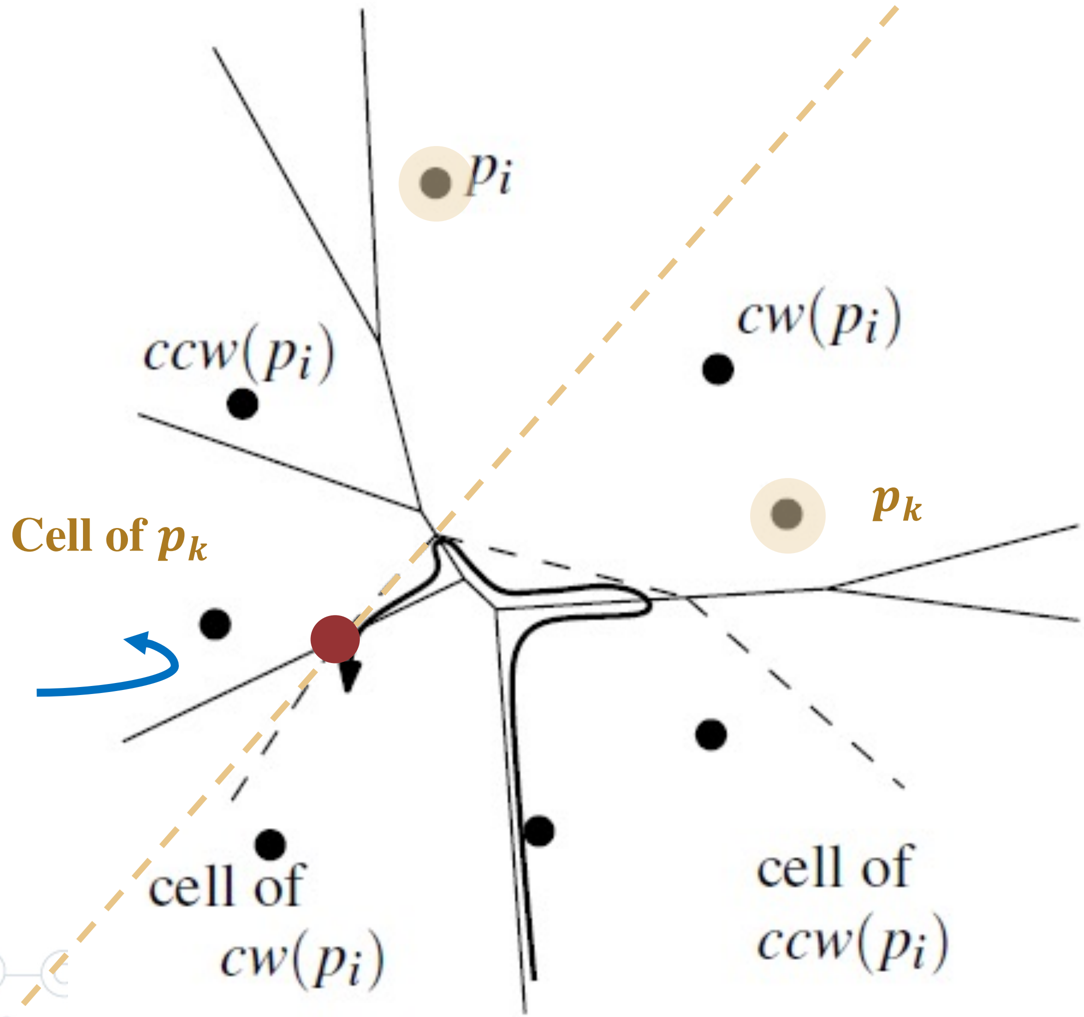
Farthest point Voronoi diagram



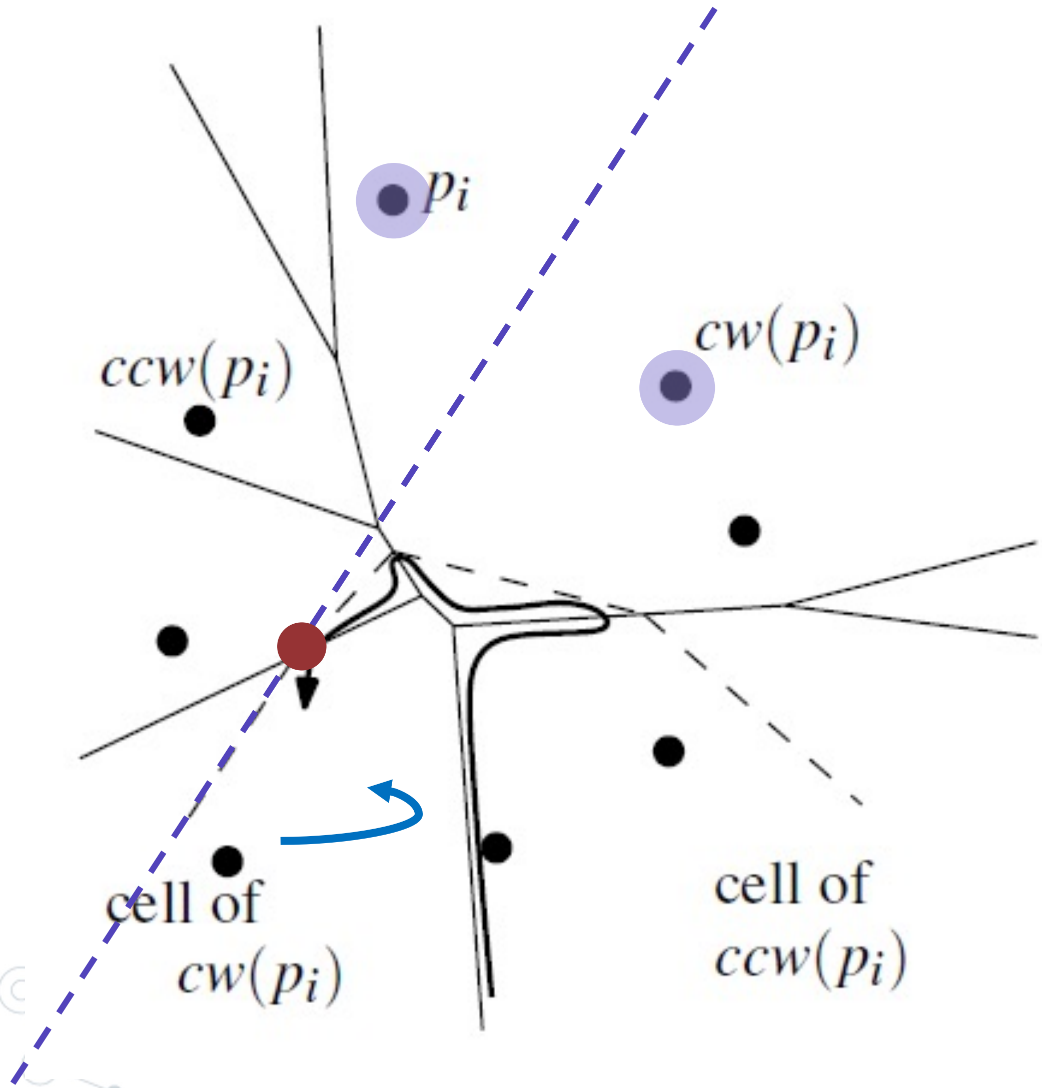
Farthest point Voronoi diagram



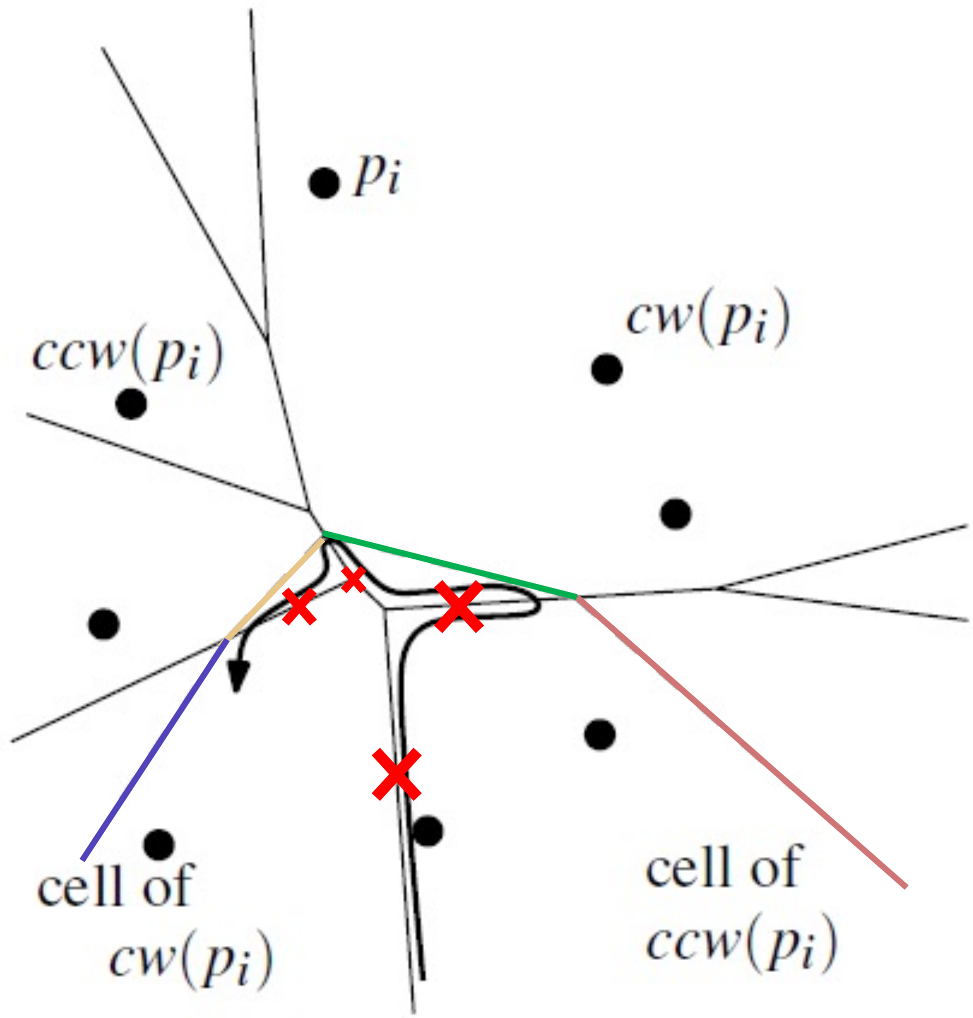
Farthest point Voronoi diagram



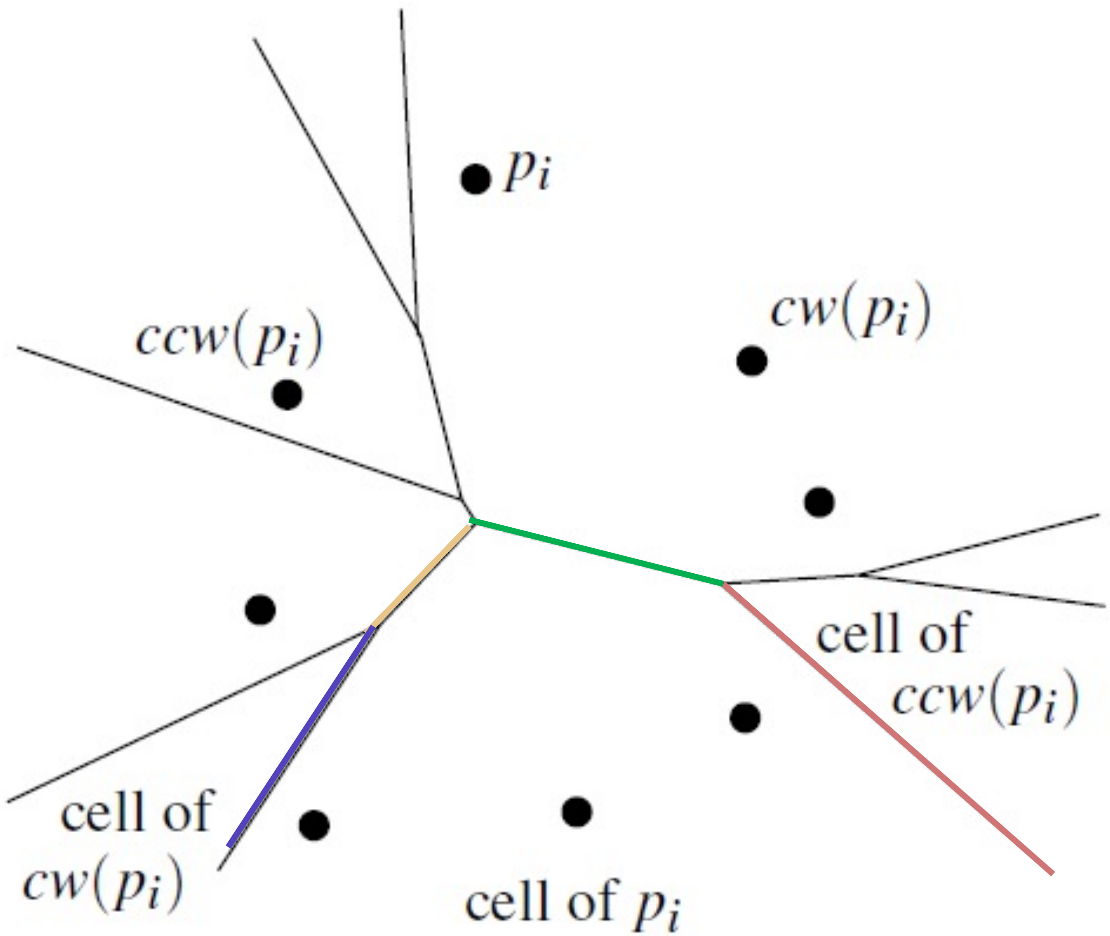
Farthest point Voronoi diagram



Farthest point Voronoi diagram



Farthest point Voronoi diagram



Farthest point Voronoi diagram

Complexity:

- CH - $O(n \log n)$
- Insertion of p_i : worst case O(i)
  Expected: O(1)
- Proof:
- The complexity of the ith insertion is as the complexity of the cell of p_i
- There are at most $2i - 3$ edges after the ith insertion

⇒ The average cell complexity is $O(1)$

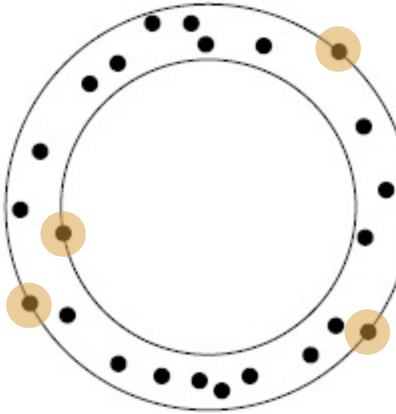
- Each point from $p_1, \ldots, p_i$ have the same probability to be the last one added ⇒ the expected complexity of insertion is $O(1)$
- **Corollary: the expected complexity is $O(n \log n)$ and the worst-case complexity is $O(n^2)$.**

Back to the smallest width ring

- Case 1: the center is a vertex of the farthest point Voronoi diagram

1. Compute the farthest point Voronoi diagram
2. For each vertex of the farthest-point Voronoi diagram:
   2.1. Determine the point of P that is closest

$O(n)$ – compute the smallest width ring. (Case 1)



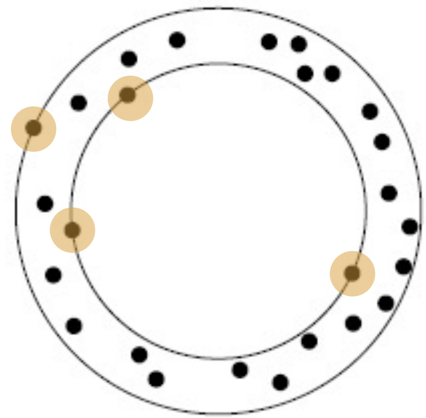
Case 1:
3 outer 1 inner

Back to the smallest width ring

- Case 2: the center is a vertex of the closest point Voronoi diagram

1. Compute the normal Voronoi diagram
2. For each vertex of the normal Voronoi diagram:
   2.1. Determine the point of P that is farthest

$O(n)$ – compute the smallest width ring. (Case 2)



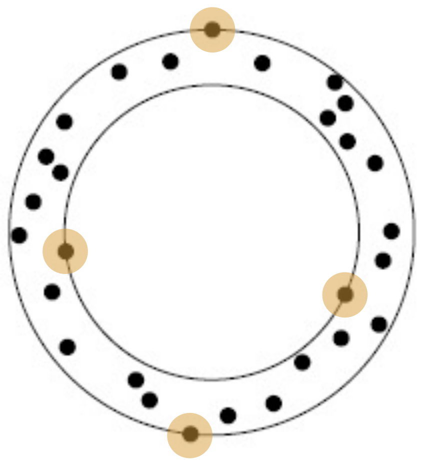
Case 2:
1 outer 3 inner

Back to the smallest width ring

- Case 3: the center is an intersection of two edges from both diagrams

1. Compute the normal Voronoi diagram and farthest point Voronoi diagram
2. For every pair of edges (one from each diagram)
   2.1. check if they intersect

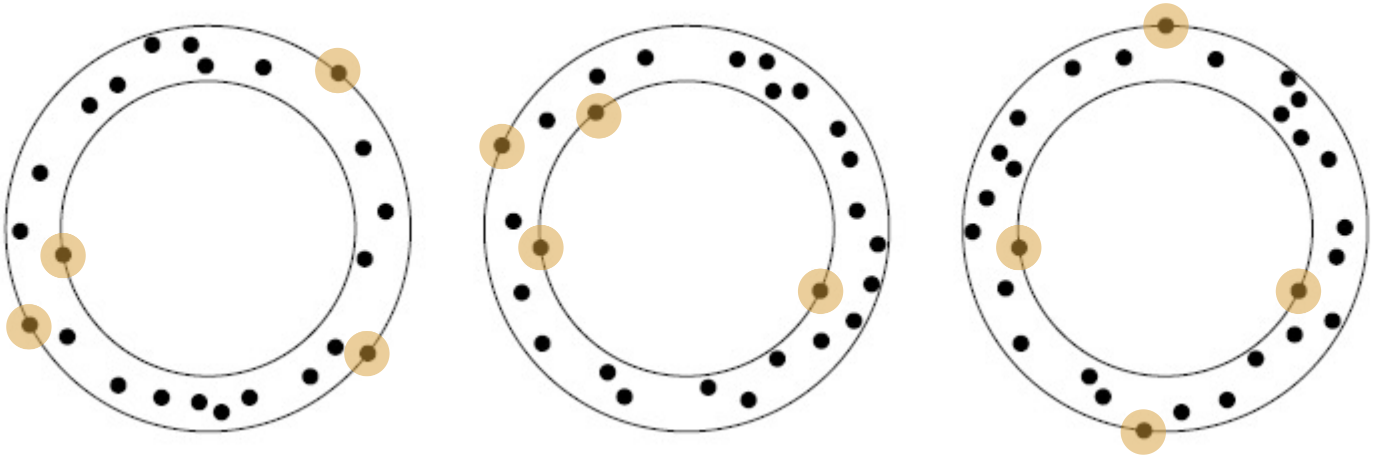
$O(n^2)$ – compute the smallest width ring. (Case 3)



Case 3:
2 outer 2 inner

Back to the smallest width ring

- Case 1: the center is a vertex of the farthest point Voronoi diagram
- Case 2: the center is a vertex of the closest point Voronoi diagram
- Case 3: the center is an intersection of two edges from both diagrams.



Case 1:
3 outer 1 inner

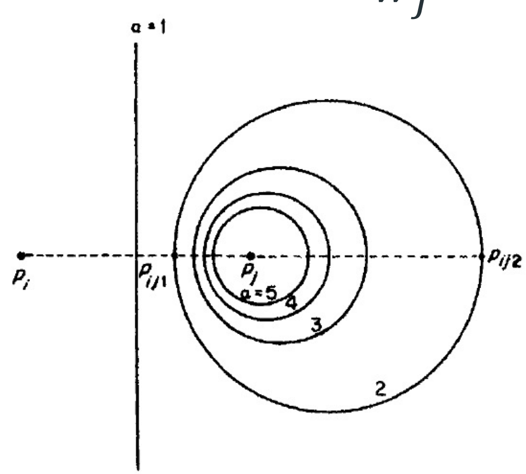
Case 2:
1 outer 3 inner

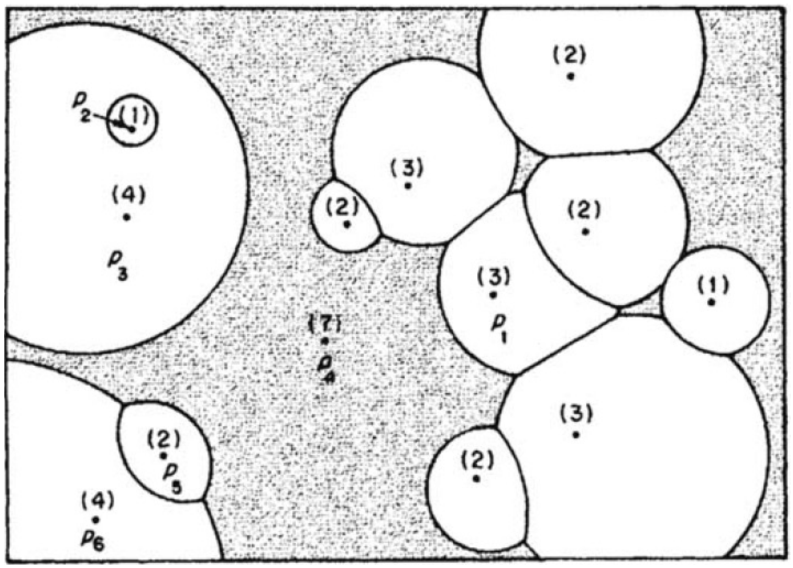
Case 3:
2 outer 2 inner

Multiplicatively Weighted Voronoi Diagram

- **Difference** – Euclidean distance between points is divided by positive weights
  - **Distance** - $\mathrm{dist}(p, s_i) = \frac{\|p - s_i\|}{w_i}$.
- Edges – circular arcs or straight-line segments
  - For every point x on the edge separating $V(s_i)$ and $V(s_j)$,

$$\mathrm{dist}(x, s_i) = \mathrm{dist}(x, s_j) \cdot \frac{w_i}{w_j}.$$

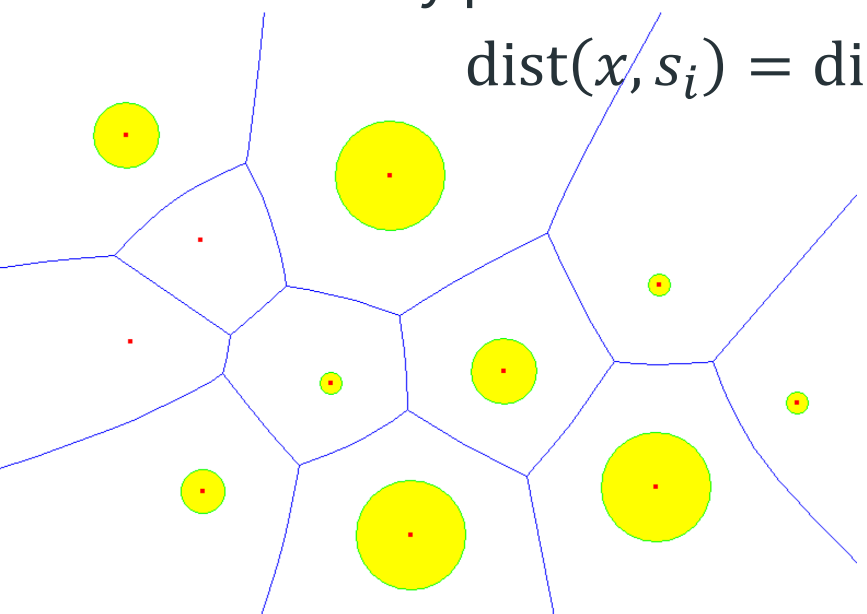


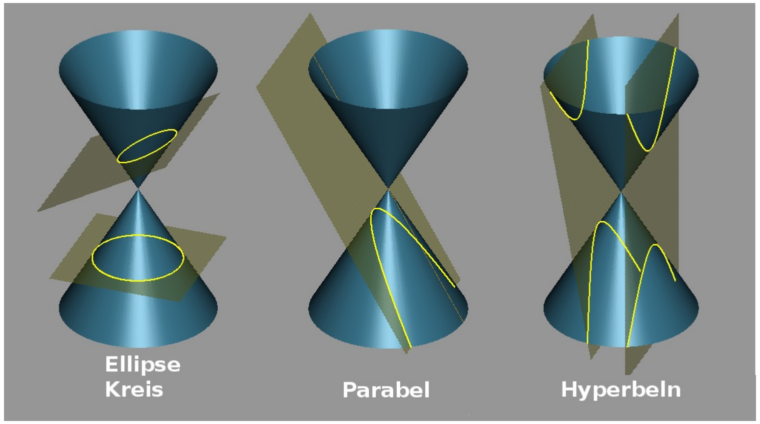


Additively Weighted Voronoi Diagram

- **Difference** – positive weights are subtracted from the Euclidean distance
  - **Distance** - $\text{dist}(p, s_i) = \|p - s_i\| - w_i$.
- Edges – hyperbolic arcs or straight-line segments
  - For every point x on the edge separating $V(s_i)$ and $V(s_j)$,

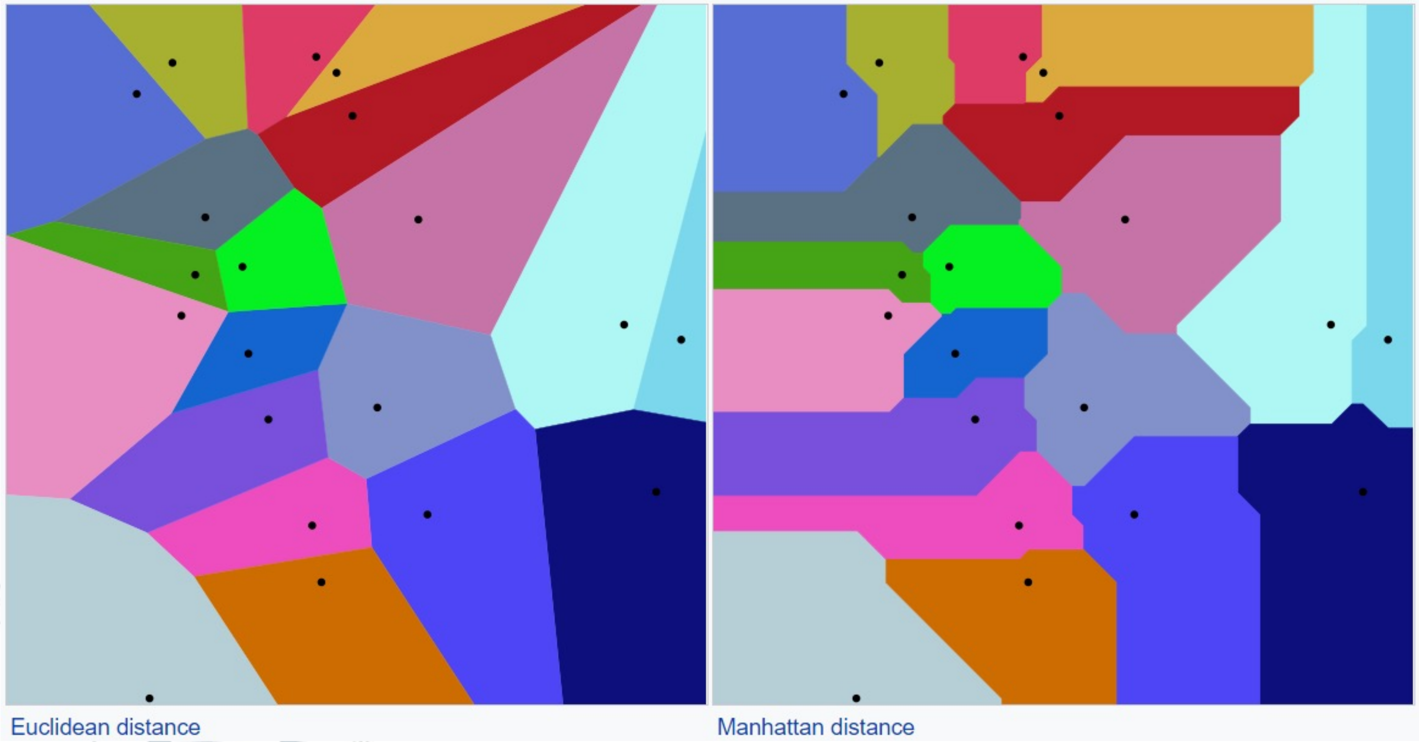
$$\text{dist}(x, s_i) = \text{dist}(x, s_j) + (w_i - w_j).$$





Voronoi Diagram in Different Metric

- **Difference** – Distance defined in L_1
  - **Distance** - $\mathrm{dist}(p, s_i) = |p_x - s_{i,x}| + |p_y - s_{i,y}|$.
- Edges – vertical, horizontal or diagonal at ± 45 degree



Centroidal Voronoi Diagram (CVD)

- **Difference** – Each site is the mass centroid of each cell
  - Given a region $V \in \mathrm{R}^N$, and a density function ρ,

    mass centroid $\boldsymbol{z}^*$ of V is defined by $\boldsymbol{z}^* = \frac{\int_V \boldsymbol{y}\rho(\boldsymbol{y})\,d\boldsymbol{y}}{\int_V \rho(\boldsymbol{y})\,d\boldsymbol{y}}$
  - **Centroid of polygon** (CCW order of the vertices (x_i, y_i))

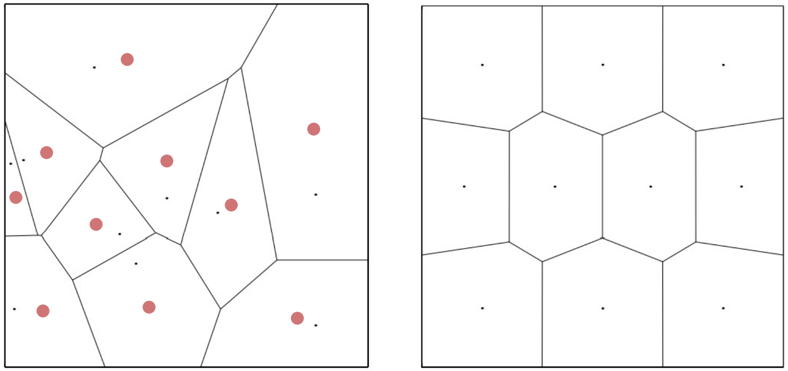
$$Area = A = \frac{1}{2}\sum_{i=0}^{N-1}(x_i y_{i+1} - x_{i+1} y_i)$$

$$x_c = \frac{1}{6A}\sum_{i=0}^{N-1}(x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$

$$y_c = \frac{1}{6A}\sum_{i=0}^{N-1}(y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$

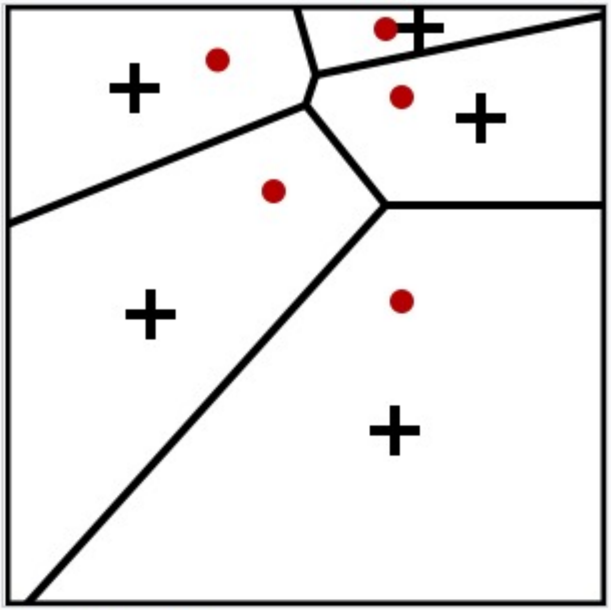
CVD Computation – Lloyd's Algorithm

1. Compute the Voronoi Diagram of the given set of sites $\{s_i\}_{i=1}^{n}$;
2. Compute the mass centroids of Voronoi cells $\{V_i\}_{i=1}^{n}$ found in step 1, these centroids are the new set of sites;
3. If this new set of sites meets the **convergence criterion**, terminate;
   Else, return to step 1.

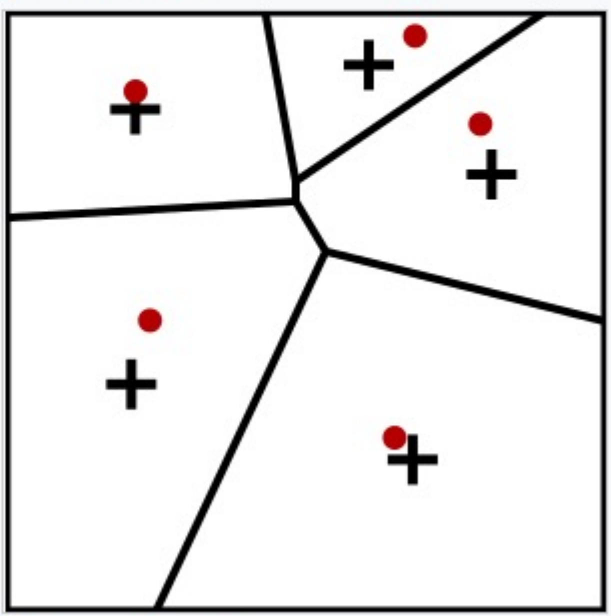


Note

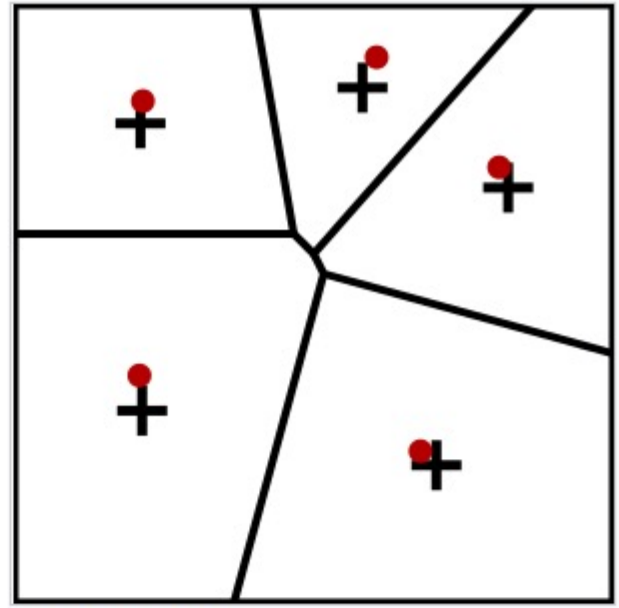
- Convergence criterion depends on specific application
- Converges to a CVD slowly, so the algorithm stops at a tolerance value
- Simple to apply and implement



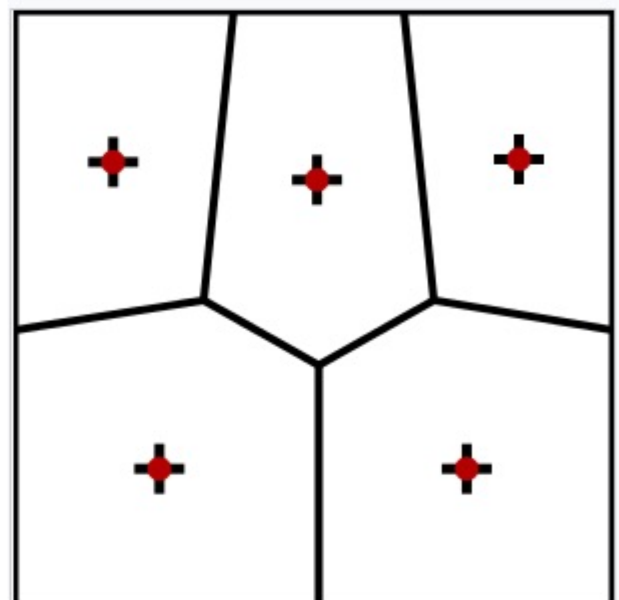
First iteration



Second iteration



Third iteration



Fifteenth iteration

Voronoi Diagram in Higher Dimensions

- **Cells** – convex polytopes
- **Bisectors** - $(d-1)$-dimensional hyper-planes
- **Complexity** - $O(n^{\lceil \frac{d}{2} \rceil})$